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STRUCTURAL REPRESENTATIONS IN THE MECHANICS OF ELASTO-PLASTIC STRAINS

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There is proposed a representation of inelastic (plastic) strain as a certain structural distortion in an elastic body. There is considered the possibility of using such an inelastic strain representation to investigate the processes of microcrack development, and to solve elasto-plastic problems. The structural imperfections and defects causing states of stress in the body, which are obtained in problems about the tension of an infinite plane with a hole in /1,2/, are determined.

1. Fundamental statements. Let a certain structural distortion, defined by the components of the inelastic (plastic) strain tensor Γ_{jk} ($j, k = x, y, z$) occur in a solid. Quantitatively, the inelastic strain is determined by the difference between the total γ_{jk} and the elastic γ_{jk}^e strains, or

$$\Gamma_{jk} = \frac{\partial u_j}{\partial k} + \frac{\partial u_k}{\partial j} - \gamma_{jk}^e \tag{1.1}$$

where u_j, u_k are displacement components in the direction of the appropriate axes. It is assumed that the elastic strain is determined in terms of stress by Hooke's law, while the plastic strain is the result of slip along the interatomic planes in the elastic body (here and henceforth the repeated subscripts are omitted)

$$\Gamma_x + \Gamma_y + \Gamma_z = 0 \tag{1.2}$$

We shall henceforth consider the plane problem (all the quantities are functions of two variables x and y and $\gamma_{xz} = \gamma_{yz} = 0$), including plane strain characterized by the condition

$$\gamma_z = 0 \tag{1.3}$$

and the plane state of stress for which

$$\sigma_z = 0 \tag{1.4}$$

Expressing the stress tensor by Hooke's law in terms of the elastic strain tensor, and taking (1.1) into account, the equilibrium equations in displacements can be represented for the plane problem in such a form (G is the shear modulus) /3/:

$$G\Delta u_x + \frac{2G}{(\varkappa-1)}\frac{\partial}{\partial x}\left(\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y}\right) + \rho_x = 0 \tag{1.5}$$

$$G\Delta u_y + \frac{2G}{(\varkappa-1)}\frac{\partial}{\partial y}\left(\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y}\right) + \rho_y = 0; \quad \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

For plane strain

$$\rho_x = -G\left(\frac{\partial\Gamma_x}{\partial x} + \frac{\partial\Gamma_{xy}}{\partial y}\right), \quad \rho_y = -G\left(\frac{\partial\Gamma_y}{\partial y} + \frac{\partial\Gamma_{xy}}{\partial x}\right), \quad \varkappa = 3-4\nu \tag{1.6}$$

where ν is the Poisson's ratio, and in the case of the plane state of stress

$$\rho_x = -G\left(\frac{\partial\Gamma_x}{\partial x} + \frac{\partial\Gamma_{xy}}{\partial y}\right) + \frac{G\nu}{1-\nu}\frac{\partial\Gamma_z}{\partial x} \tag{1.7}$$

$$\rho_y = -G\left(\frac{\partial\Gamma_y}{\partial y} + \frac{\partial\Gamma_{xy}}{\partial x}\right) + \frac{G\nu}{1-\nu}\frac{\partial\Gamma_z}{\partial y}, \quad \varkappa = \frac{3-\nu}{1+\nu}$$

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The following forces, referred to unit length of boundary (n is the outer normal to the boundary) must be added to the external loads on the boundary L of the plastic zone:
For plane strain

$$X_L = G\,[\Gamma_x \cos(nx) + \Gamma_{xy}\cos(ny)]$$
$$Y_L = G\,[\Gamma_y \cos(ny) + \Gamma_{xy}\cos(nx)] \quad (x,\ y \in L) \qquad .8)$$

For the plane state of stress

$$X_L = G\left[\left(\Gamma_x - \frac{\nu}{1-\nu}\Gamma_z\right)\cos(nx) + \Gamma_{xy}\cos(ny)\right]$$
$$Y_L = G\left[\left(\Gamma_y - \frac{\nu}{1-\nu}\right)\Gamma_z\cos(ny) + \Gamma_{xy}\cos(nx)\right] \quad (x, y \in L) \qquad (1.9)$$

It follows from equations (1.5) that the problem of determining the displacements for a given inelastic strain (1.1) reduces /3/ to the plane problem of the theory of elasticity with additional (fictitious) mass (1.5) or (1.7) and surface loads (1.8) or (1.9).

2. **Direct Structural Problem.** Let us determine the state of stress caused by the given strains (1.1). This state of stress is related by Hook's law to the elastic strains and can be represented as the difference in the fictitious stress components σ_{jk}° (due to the fictitious loads), and the stress calculated by Hooke's law in terms of the inelastic strains /3/. It follows from the above that for plane strain there will be

$$\sigma_x + \sigma_y = \sigma_x^{\circ} + \sigma_y^{\circ} - G\,(\Gamma_x + \Gamma_y)$$
$$\sigma_y - \sigma_x + 2i\sigma_{xy} = \sigma_y^{\circ} - \sigma_x^{\circ} + 2i\sigma_{xy}^{\circ} - G\,(\Gamma_y - \Gamma_x + 2i\Gamma_{xy}) \qquad (2.1)$$

and for the plane stress state there will be

$$\sigma_x + \sigma_y = \sigma_x^{\circ} + \sigma_y^{\circ} - G\,\frac{1+\nu}{1-\nu}\,(\Gamma_x + \Gamma_y) \qquad (2.2)$$

The second relationship remains the same as for plane strain.

We shall still consider the body unbounded and the inelastic strain tensor components zero on the boundary of the slip domain. Then the Kolosov—Muskhelishvili functions for the fictitious stresses are found /4/ as integrals over the slip domain D, of the stress functions corresponding to a lumped force applied at an arbitrary point $z_0 = x_0 + iy_0$ of the infinite plane

$$\Phi_*(z) = -\iint\limits_D \frac{A}{z-z_0}\,dx_0dy_0, \quad \Psi_*(z) = \iint\limits_D \left[\frac{\varkappa\bar{A}}{z-z_0} - \frac{A\bar{z}_0}{(z-z_0)^2}\right] dx_0dy_0, \quad A = \frac{\rho_x + i\rho_y}{2\pi(1+\varkappa)} \qquad (2.3)$$

The continuous function, twice-differentiable in the domain, which is zero on the boundary can be written as follows

$$\Gamma(x, y) = -\frac{1}{4\pi}\int\limits_L \frac{\partial\Gamma}{\partial n}\ln r^2\,dl + \frac{1}{4\pi}\iint\limits_D \Delta\Gamma \ln r^2\,dx_0dy_0 \qquad (2.4)$$
$$r^2 = (x-x_0)^2 + (y-y_0)^2$$

Here n is the external normal to the line L and l is the length of this line which characterizes the location of the point (x_0, y_0).

Representing the last terms in (2.1) and (2.2) in the form of integrals by means of (2.4), and integrating by parts in (2.3), we obtain the stress functions due to the structural distortion (transformation) in the form:

$$\Phi(z) = -\frac{G}{\pi(1+\varkappa)}\left[\iint\limits_D p(x_0, y_0)\ln(z-z_0)\,dx_0dy_0 + \int\limits_L p_L(l)\ln(z-z_0)\,dl\right] \qquad (2.5)$$
$$\Psi(z) = \frac{G}{\pi(1+\varkappa)}\left[\iint\limits_D \frac{p(x_0, y_0)\,\bar{z}_0}{z-z_0}\,dx_0dy_0 + \int\limits_L \frac{p_L(l)\,\bar{z}_0}{z-z_0}\,dl\right]$$
$$p(x_0, y_0) = \frac{1}{2}\frac{\partial^2\Gamma_x}{\partial y_0^2} + \frac{1}{2}\frac{\partial^2\Gamma_y}{\partial x_0^2} - \frac{\partial^2\Gamma_{xy}}{\partial x_0\partial y_0} + \chi \qquad (2.6)$$
$$p_L(l) = -\frac{1}{2}\left[\left(\frac{\partial\Gamma_y}{\partial x_0} - \frac{\partial\Gamma_{xy}}{\partial y_0}\right)\cos(nx_0) + \left(\frac{\partial\Gamma_x}{\partial y_0} - \frac{\partial\Gamma_{xy}}{\partial x_0}\right)\cos(ny_0)\right] + \chi_L \qquad (2.7)$$

where for the plane state of stress

$$\chi = 0,\ \chi_L = 0$$

and for plane strain

$$\chi = \frac{\nu}{2}\Delta\Gamma_z,\quad \chi_L = -\frac{\nu}{2}\frac{\partial\Gamma_z}{\partial n}$$

The quantity $p(x_0, y_0)$ is the component of the strain incompatibility tensor /5/ for the plane state of stress and for plane-plastic strain ($\Gamma_z \equiv 0$).

From a comparison of the expressions (2.5) obtained and the state of stress /6/ of a wedge dislocation in an infinite plane, it is seen that the state of stress caused by the structural transformation can be represented as the stress from wedge dislocations distrubuted over the domain D with the density $p(x_0, y_0)$ and over the boundary of the domain with the density $p_L(l)$.

To find the state of stress occuring from the structural transformation made in an arbitrary simply-connected body, it is evidently necessary to determine the stress from a wedge dislocation in this body, and then to evaluate the appropriate integrals over the slip domain and its boundary. A doubly-connected body is examined below (Sect.4).

3. **Inverse structural problem.** We designate the determination of the inelastic strains causing a known state of stress as such a problem. It is solved in two stages. First is the calculation of the structural imperfections (dislocations) in given stresses, and second is the determination of the inelastic strains in the imperfections found. In general the second stage does not have an unique solution /3/. To predefine the problem, the history of the origin of the state of stress must be known, which will permit finding the slip line at each instant, and distributing the imperfections over the domain D; in principle, the inelastic strain may thereby be found.

The problem is solved sufficiently simply in the first stage if it is conceived that the strains in whose terms the densities of the wedge-type dislocations are expressed, $p(x_0, y_0)$ and $p_L(l)$, are obtained during unloading, i.e., they are determined by Hooke's law in terms of the given stress components.

Then, taking account of conditions (1.3) and (1.4), we find from (2.6)

$$p(x_0 y_0) = -\frac{1+\varkappa}{8G}\Delta(\sigma_x + \sigma_y) \tag{3.1}$$

Here $\varkappa$ is determined by the last formulas in (1.6) and (1.7), respectively, for the plane strain and the plane state of stress.

To find the density of the wedge-type dislocations distributed over the boundary L of the slip domain, the derivatives of the strains in (2.7) must be evaluated during unloading, in terms of discontinuities in the derivatives of the stress components on the line L

$$p_L(l) = -\frac{1+\varkappa}{8G}\left[\frac{\partial(\sigma_x + \sigma_y)}{\partial n}\right] \tag{3.2}$$

The square brackets denote discontinuities of the quantities within them on the slip domain boundary; it is calculated upon going from points within the domain to points outside the domain. Let us note that the equilibrium equations were used in obtaining (3.1) and (3.2).

4. **Examples.** In plane with a circular hole (of radius R), let the yield point be reached at infinity under the effect of an axisymmetric load. Then an additional loading occurs such that the stress at infinity is $\sigma_x = q_1,\ \sigma_y = q_2$. It is assumed that the plasticity domain will be expanded monotonically. For such additional charges we consider two ideal plasticity problems in which the resistance to shear /7/ is considered a constant equal to the yield point (τ_T).

L.A. Galin problem. In a polar coordinate system, the stress components in a plasticity zone are:

$$\sigma_r = 2\tau_T \ln\frac{r}{R},\quad \sigma_\varphi = 2\tau_T\left(1 + \ln\frac{r}{R}\right),\quad \tau_{r\varphi} = 0 \tag{4.1}$$

to which the complex potentials

$$\Phi_1(z) = \tau_T\left(\ln\frac{z}{R} + \frac{1}{2}\right),\quad \Psi_1(z) = 0 \tag{4.2}$$

correspond.

The Muskhelishvili stress functions in the elastic domain will be /1/

$$\Phi_2(z)=\frac{q_1+q_2}{4}-\tau_T\ln\left(\frac{1}{2}+\frac{1}{2z}\sqrt{z^2-4\lambda b^2}\right) \quad (4.3)$$
$$\Psi_2(z)=\tau_T\left[\lambda+\frac{1+\lambda^2}{2\lambda}\left(\frac{1}{\sqrt{z^2-4\lambda b^2}}-1\right)\right]$$
$$\lambda=\frac{q_2-q_1}{2\tau_T},\quad b=R\exp\left(\frac{q_1+q_2}{4\tau_T}-\frac{1}{2}\right)$$

The boundary L between the domains is an ellipse whose exterior is mapped on the exterior of a unit circle by the function

$$z=b\,(\zeta+\lambda/\zeta) \quad (4.4)$$

The solution presented is valid while the maximal tangential stress is in areas perpendicular to the plane of load action. For the majority of materials $(0.5\geqslant\nu\geqslant 0.3)$, this condition is satisfied for $b\,(1+\lambda)/R\leqslant 2.5$.

The sum of the normal stresses (4.1) is a harmonic function. It then follows from (3.1) that there are no structural imperfections distributed over the plasticity domain, i.e.

$$p\,(x_0,y_0)=0.$$

The derivative of the sum of the normal stresses is discontinuous on the boundary L of the slip domain. This means that wedge-type dislocations are introduced, whose density can be found from (3.2)

$$p_L(l)=\frac{2\,(1-\nu)\,\tau_T}{G}\frac{\partial\,|\zeta|}{\partial n} \quad (4.5)$$

Let us make a slit from the contour of the circular hole to infinity. Then the relative displacement and the rotation of the sections at the site of the slit will be determined by the stress (elastic strain) components on the contour itself.

That relative displacement of the sections for which a ring dislocation with the angle of divergence

$$\alpha=-4\pi\tau_T\,(1-\nu)/G \quad (4.6)$$

is formed, will evidently correspond to the state of stress (4.1).

Therefore, structural imperfections (wedge-type dislocations) and defects (ring-like dislocations) are obtained which occur in a plane with a circular hole in the deformation process considered.

Theorem. The state of stress in the L.A. Galin problem is the state of stress occuring in an elastic plane with a hole due to an external load, structural imperfections (4.5), and the ring dislocation (4.6).

For the proof we evaluate the state of stress due to each of the factors.

The Muskhelishvili functions due to the external loads are /4/

$$\Phi_q(z)=\frac{(q_2-q_1)\,R^2}{2z^2}+\frac{q_1+q_2}{4},\qquad \Psi_q(z)=\frac{q_2-q_1}{2}\left(1+\frac{3R^4}{z^4}\right)+\frac{(q_1+q_2)\,R^2}{2z^2} \quad (4.7)$$

To evaluate the state of stress due ot the structural imperfections, we represent the stress functions for a wedge dislocation of power ε inserted at an arbitrary point (z_0) of the infinite plane, in the form /6/:

$$\Phi(z)=-\frac{\varepsilon G}{4\pi(1-\nu)}\ln(z-z_0),\quad \Psi(z)=\frac{\varepsilon G\bar z_0}{4\pi(1-\nu)(z-z_0)} \quad (4.8)$$

These functions are determined to the accuracy of all-around tension at infinity. Using them, we obtain the Muskhelishvili functions for a wedge-type dislocation placed at an arbitrary point of the infinite plane with circular hole of radius R

$$\Phi_R(z)=\frac{\varepsilon G}{4\pi(1-\nu)}f_1(z,z_0),\quad \Psi_R(z)=\frac{\varepsilon G}{4\pi(1-\nu)}f_2(z,z_0) \quad (4.9)$$
$$f_1(z,z_0)=\ln\frac{R^2-z\bar z_0}{z\bar z_0(z_0-z)R}-\frac{R^2(z-z_0)}{z(R^2-z\bar z_0)}$$
$$f_2(z,z_0)=\frac{\bar z_0}{z-z_0}-\frac{R^2}{z^2}\ln z_0\bar z_0+\frac{2R^2z_0}{z^3}+\frac{R^2z_0(R^2z_0+R^2z-2zz_0\bar z_0)}{z^2(R^2-z\bar z_0)^2}$$

The state of stress from the ring-like dislocation is determined as the stress from a wedge-type dislocation in a plane with a circular hole at its center

$$\Phi_k = \tau_T \ln z, \quad \Psi_k(z) = \tau_T \frac{R^2}{z^2}(2\ln R - 1) \tag{4.10}$$

Keeping in mind that the boundary L between the plastic domain and the elastic zone goes over into the unit circle γ by the transformation (4.4), the state of stress from the wedge-type dislocations inserted on this boundary can be written in the form of an integral over the contour γ. Taking account of (4.5) and (4.9), we obtain

$$\Phi_L(z) = \frac{\tau_T}{2\pi i}\int_\gamma f_1[z, \omega(\zeta_0)]\frac{d\zeta_0}{\zeta_0}, \quad \Psi_L(z) = \frac{\tau_T}{2\pi i}\int_\gamma f_2[z, \omega(\zeta_0)]\frac{d\zeta_0}{\zeta_0} \tag{4.11}$$

Evaluating the last integrals, we find the stress functions due to wedge dislocations distributed on the boundary

In the plastic domain $(|z + \sqrt{z^2 - 4\lambda b^2}| \leqslant 2b)$

$$\Phi_L(z) = -\tau_T \ln b - \frac{(q_2 - q_1)R^2}{2z^2} \tag{4.12}$$
$$\Psi_L(z) = 2\tau_T \frac{R^2}{z^2}\ln b - \frac{q_2 - q_1}{2}\left(1 + \frac{3R^4}{z^4}\right)$$

In the elastic domain $(|z + \sqrt{z^2 - 4\lambda b^2}| \geqslant 2b)$

$$\Phi_L(z) = -\frac{\tau_T}{2}\ln(z + \sqrt{z^2 - 4\lambda b^2}) + \frac{(q_2 - q_1)R^2}{2z^2} \tag{4.13}$$
$$\Psi_L(z) = \tau_T\left[\frac{1+\lambda^2}{2\lambda}\left(\frac{z}{\sqrt{z^2 - 4\lambda b^2}} - 1\right) - \frac{3\lambda R^4}{z^4} - \frac{R^2}{z^2}(2\ln R - 1)\right] - \frac{(q_1 + q_2)R^2}{2z^2}$$

It can be confirmed that

$$\Phi_{1,2}(z) = \Phi_q(z) + \Phi_k(z) + \Phi_L(z) \tag{4.14}$$
$$\Psi_{1,2}(z) = \Psi_q(z) + \Psi_k(z) + \Psi_L(z)$$

This means the theorem is proved.

G.P. Cherepanov problem /2/. A thin plate with a circular hole, stretched beyond the elastic limit by the loads considered above, is considered. The stress components in the plastic domain will be

$$\sigma_r = 2\tau_T\left(1 - \frac{R}{r}\right), \quad \sigma_\varphi = 2\tau_T, \quad \tau_{r\varphi} = 0 \tag{4.15}$$

An oval, whose exterior is mapped conformally on the exterior of a unit circle by the function

$$\omega(\zeta) = \frac{R(2\zeta^2 + a)^2}{c(a^2 - 4)\zeta^3}, \quad c = \frac{q_1 + q_2 - 4\tau_T}{2\tau_T} \tag{4.16}$$

is the boundary between the plastic domain and the elastic zone. The parameter a is the real root of the cubic equation

$$a^3 + 4a + \frac{8(q_2 - q_1)}{q_1 + q_2 - 4\tau_T} = 0$$

The Muskhelishvili functions in the elastic zone are

$$\Phi(\zeta) = 2\tau_T - \frac{q_1 + q_2}{4} + \frac{2c\tau_T\zeta^2}{2\zeta^2 + a} \tag{4.17}$$
$$\Psi(\zeta) = -ac\tau_T\frac{\zeta^4(1 + \zeta^2)[2(a^2 + 4)\zeta^2 - a(4 - 3a^2)]}{(2\zeta^2 + a)^3(2\zeta^2 - 3a)}$$

For unloading not to occur in the plastic domain, the parameters a and c should satisfy the conditions $0 < a < 2/3$, $-1 < c < -0.5$.

The sum of the normal stresses (4.15) is not a harmonic function in this case, and the density of the wedge-type dislocations distributed over the domain does not equal zero

$$p(r) = \frac{\tau_T R}{(1 + \nu)Gr^3} \tag{4.18}$$

The density of the structural imperfections distributed over the boundary of the elastic and plastic zones is found analogously to the previous problem as

$$p_L(l) = -\frac{c\tau_T(4+3a^2)}{(1+\nu)G[4+a^2+2a(\zeta+\bar{\zeta})]}\frac{\partial|\zeta|}{\partial n} \qquad (4.19)$$

In addition to the imperfections found in the problem under consideration, there is, exactly as for plane strain, a structural defect in the form of a ring-type dislocation with the parameter

$$\alpha = -\frac{2\pi\tau_T}{(1+\nu)G} \qquad (4.20)$$

Just as had been done in the L.A. Galin problem, it can be shown that the state of stress (4.15) and (4.17) obtained by G.P. Cherepanov in solving the elasto-plastic problem is the state of stress occurring in an elastic body due to the structural imperfections (4.18) and (4.19), the defect (4.20), and the given external load.

Structural imperfections have a finite magnitude in a real solid and are lumped in quite small volumes. The kernels of these imperfections are sharp stress concentrators and microcracks are actually always formed ahead of them. Material rupture is evidently associated with this process.

A discrete model must be introduced for a strict examination of fracture processes. However, in any cases of developed plastic strain, the density of continuously distributed structural distortions can also be a characteristic of the strength of the material.

Representation of the plastic strain of a solid in the form of a structural transformation permits its consideration as a single elastic body in which structural distortions occur, when solving boundary value problems of the mechanics of inelastic strains. There is here no need to separate the body into elastic and plastic parts with unknown boundaries, which affords the possibility of a more simple and strict formulation of the elasto-plastic problem.

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